Vasicek Model

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The **Vasicek model** (one factor model describing the evolution of interest rate) is given by

$$dr_t = a\,(b - r_t)\,dt + \sigma dW_t$$

where w_t is a Wiener process. Notice that coefficient a determines the speed of revision, b determines the long term average and σ is the instantaneous volatility. To see this, applying Itô's lemma to $e^{at}r_t$, we can get

$$\begin{aligned} d\left(e^{at}r_t\right) &= ae^{at}r_t dt + e^{at}dr_t \\ &= \left(ae^{at}r_t + ae^{at}\,(b - r_t)\right)dt + e^{at}\sigma dW_t \end{aligned}$$

Integrate both ends from 0 to T, we can arrive at the conclusion that

$$e^{aT}r_T - r_0 = b\left(e^{aT} - 1\right) + \sigma \int_0^T e^{at}dW_t$$

thus

$$r_T = r_0 e^{-aT} + b\left(1 - e^{-aT}\right) + \sigma e^{-aT}\int_0^T e^{at}dW_t$$

The mean of r_T is given by

$$\mathbb{E}\left[r_T\right] = r_0 e^{-aT} + b\left(1 - e^{-aT}\right)$$

and

$$\lim_{T\to\infty} \mathbb{E}\left[r_T\right] = b$$

The variance of r_T is represented by

$$\begin{aligned} var\left[r_T\right] &= \left(\sigma e^{-aT}\right)^2 \mathbb{E}\left[\left(\int_0^T e^{at}dW_t\right)^2\right] \\ &= \sigma^2 e^{-2aT}\int_0^T e^{2at}dt \\ &= \frac{\sigma^2}{2a}\left(1 - e^{-2aT}\right) \end{aligned}$$

and

$$\lim_{T\to\infty} var\left[r_T\right] = \frac{\sigma^2}{2a}$$